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On weak KAM theory for N-body problems

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Abstract. We consider N-body problems with potential $1/r^{2\kappa}$, where $\kappa \in (0, 1)$, including the Newtonian case ($\kappa = 1/2$). Given $R > 0$ and $T > 0$, we find a uniform upper bound for the minimal action of paths binding, in time T, any two configurations which are contained in some ball of radius R. Using cluster partitions, we obtain from these estimates the Hölder regularity of the critical action potential (i.e. of the minimal action of paths binding two configurations in free time). As an application, we establish the weak KAM theorem for these N-body problems, i.e. we prove the existence of fixed points of the Lax–Oleinik semigroup, and we show that they are global viscosity solutions of the corresponding Hamilton–Jacobi equation. We also prove that there are invariant solutions for the action of isometries on the configuration space.

1. *Introduction*

Let E be a finite dimensional Euclidian space, and denote by $x = (r_1, \ldots, r_N) \in E^N$ the configuration vector of N punctual masses $m_1, \ldots, m_N > 0$. By $\|x\|$, we will denote the norm given by $\max\{\|r_i\|_E \mid 1 \leq i \leq N\}$, and $|x|$ will denote the norm induced by the mass scalar product

$$\langle x, y\rangle = \sum_{i=1}^{N} m_i \langle r_i, s_i\rangle_E$$

for $x = (r_1, \ldots, r_N)$, $y = (s_1, \ldots, s_N) \in E^n$. As usual, we call $I(x) = |x|^2$ the moment of inertia of x regarding the origin of E. The N-body problem is determined once the force function U on E^N (or the potential function), the negative of the potential energy, has been chosen. In this paper, we restrict ourselves to potential functions which are homogeneous of degree -2κ,

$$U_\kappa(x) = \sum_{i<j} m_i m_j (r_{ij})^{-2\kappa},$$

where $r_{ij} = \|r_i - r_j\|_E$, and $\kappa \in (0, 1)$. The case $\kappa = 1/2$ corresponds to the Newtonian potential. In other words, this means that the laws of motion are given on the open and dense subset $\Omega = \{x \in E^N \mid U_\kappa(x) < +\infty\}$ by the differential equation $\ddot{x} = \nabla U_\kappa$, where the gradient is taken with respect to the mass scalar product on E^N. The equivalent variational formulation is given by the Lagrangian defined on $TE^N = E^N \times E^N$,

$$L(x, v) = \frac{1}{2} \sum_{i=1}^{N} m_i v_i^2 + U_\kappa(x),$$

where $v = (v_1, \ldots, v_n)$. Thus, motions are characterized as critical points of the Lagrangian action $A(\gamma) = \int L(\gamma(s), \dot\gamma(s))\, ds$, and the Euler–Lagrange equations define a—non-complete—analytical flow on the non-compact manifold $T\Omega$.

1.1. *Globally minimizing curves and the action potential.* Let us give a precise definition of the Lagrangian action functional. Recall that a curve $\gamma : [a, b] \to E^N$ is absolutely continuous if it is differentiable almost everywhere, and its derivative $\dot\gamma$ satisfies the fundamental theorem of calculus for the Lebesgue integral. Thus, the Lagrangian action is well defined on the set of absolutely continuous curves $\mathcal{C}$. More precisely, the action is the function $A : \mathcal{C} \to (0, +\infty]$ given by

$$A(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds = \frac{1}{2} \int_a^b |\dot\gamma(s)|^2\, ds + \int_a^b U_\kappa(\gamma(s))\, ds,$$

where $|v|$ is the norm in E^N induced by the mass scalar product. It can be seen that absolutely continuous curves with finite action are necessarily 1/2-Hölder continuous, hence they are contained in the Sobolev space $H^1([a, b], E^N)$.

For $T > 0$ and $x, y \in E^N$, denote by $\mathcal{C}(x, y, T)$ the set of all absolutely continuous curves $\gamma : [0, T] \to E^N$ which satisfy $\gamma(0) = x$ and $\gamma(T) = y$. We are interested in the function ϕ defined on $E^N \times E^N \times (0, +\infty)$ by

$$\phi(x, y, T) = \inf\{A(\gamma) \mid \gamma \in \mathcal{C}(x, y, T)\}.$$

We will say that a curve $\gamma : [a, b] \to E^N$ is globally minimizing if we have that $A(\gamma) = \phi(\gamma(a), \gamma(b), b - a)$. For a curve defined on a non-compact interval, globally minimizing will mean that the property is satisfied for all restrictions of the curve to a compact interval. It is not difficult to see that a globally minimizing curve always exists for any two configurations $x, y \in E^N$ and for all $T > 0$. Essentially, this is a consequence of the lower semicontinuity of the action functional.

In recent years, global variational methods have been successful in proving the existence of a great variety of particular motions. A typical example is the eight choreography of Chenciner and Montgomery [**4**], among many other closed orbits with topological or symmetry constraints. The main difficulty that arises from these methods for the Newtonian potential, and also for the homogeneous potentials here considered, is assuring that global minimizers avoid collisions, that is that they are contained in the open domain Ω. Following an idea of Marchal, Chenciner established a proof of this fact for the Newtonian N-body problem in a plane or a three-dimensional space (see [**3, 15**]). Simultaneously and independently, Ferrario and Terracini gave an improved version of Marchal's theorem (see [**12**]). We will not use this result anywhere in this paper, but it is relevant to remark that combined with Proposition 15 below, and using the results from [**7**], we can deduce (for the Newtonian case in dimensions greater than one) the existence of completely parabolic motions with arbitrary initial configurations. Recently, this last result was improved in [**14**].

Our first result gives an upper bound for the action of such curves; this bound depends on the size of the configurations. In our opinion, this result is fundamental for global variational methods, and it is optimal, in the sense that the bound is reached by homothetic minimizing configurations, as we explain in the following section.

THEOREM 1. *There are positive constants α, $\beta > 0$ such that for all $T > 0$,*

$$\phi(x, y, T) \leq \alpha T^{-1} R^2 + \beta T R^{-2\kappa},$$

whenever x and y are configurations contained in a ball of radius $R > 0$ of E. The constants α and β only depend on the degree of homogeneity of the potential (-2κ), the number of bodies N, and their masses.

The next result will be useful for the study of free time minimizers, that is absolutely continuous curves which minimize the action in the set of curves $\mathcal{C}(x, y) = \bigcup_{T>0} \mathcal{C}(x, y, T)$. The Mañé's critical action potential (see for instance [**6**]), or the *action potential*, is defined in our setting on $E^N \times E^N$ by

$$\phi(x, y) = \inf\{\phi(x, y, T) \mid T > 0\} = \inf\{A(\gamma) \mid \gamma \in \mathcal{C}(x, y)\}.$$

It is clear that $\phi(x, y) = \phi(y, x)$, and that $\phi(x, y) \leq \phi(x, z) + \phi(z, y)$, for any configurations x, y, z in E^N. In fact, Proposition 6 shows that the action potential ϕ is a distance function. Notice that as a corollary of Theorem 1, we have that $\phi(x, y) \leq (\alpha + \beta) R^{1-\kappa}$, whenever x and y are configurations contained in a ball of radius $R > 0$ of E. With similar arguments as in Theorem 1, combined with a cluster decomposition, we obtain the following theorem.

THEOREM 2. *There is a positive constant $\eta > 0$ such that for all $x, y \in E^N$,*

$$\phi(x, y) \leq \eta \|x - y\|^{1-\kappa}.$$

The action potential is therefore Hölder continuous with respect to the Euclidean norm on $E^N \times E^N$. In other words, for any configurations x, y, z in E^N, we have

$$\phi(x, z) - \phi(y, z) \leq \phi(x, y) \leq \eta \|x - y\|^{1-\kappa}.$$

On the other hand, it is easy to prove that the action potential is locally Lipschitz in the open and dense subset $\Omega \times \Omega \subset E^N \times E^N$.

The action potential ϕ was introduced by Mañé in the 1990s, as well as Fathi's weak KAM theorem, for the study of the dynamics of Tonelli Lagrangians on compact manifolds. However, for the Newtonian case of our N-body problems, the corresponding action potential is nothing but the minimal action between two configurations for the Maupertuis action functional. This last can be defined as the energy functional associated with the Jacobi metric in the zero-energy level.

1.2. *On the weak KAM theory.* In order to give applications, we will show that Theorem 2 enables us to prove a weak KAM theorem in the spirit of [**10, 11**]. The novelty in this approach is that we regard the action of the Lax–Oleinik semigroup on a space of Hölder functions.

Let us remember that a function $u : E^N \to \mathbb{R}$ is said to be *dominated* by L if it satisfies the condition $u(x) - u(y) \leq \phi(x, y)$ for all $x, y \in E^N$. Since the action potential is symmetric, Theorem 2 implies that dominated functions are Hölder continuous. On the other hand, it is not difficult to prove that they are locally Lipschitz in the open subset of total measure $\Omega \subset E^N$ (see Proposition 7 below). Dominated functions are therefore differentiable almost everywhere. We will discuss this in more detail below. Another way to define the set of dominated functions is to use the Lax–Oleinik semigroup: given a function $u : E^N \to [-\infty, +\infty)$ and $t > 0$ we define $T_t^- u : E^N \to [-\infty, +\infty)$ by

$$T_t^- u(x) = \inf\{u(y) + \phi(x, y, t) \mid y \in E^N\}.$$

Then, a continuous function u is dominated if and only if $u \leq T_t^- u$ for all $t > 0$. Notice that the set of dominated functions is convex and stable under the Lax–Oleinik semigroup. Setting $T_0^- u = u$ for any function u, we will prove that $(T_t^-)_{t \geq 0}$ is a continuous semigroup on the set of dominated functions equipped with the topology of uniform convergence on compact subsets.

Another set which is stable by the Lax–Oleinik semigroup is the set of functions which are invariant by symmetries. If we observe that the group of isometries of E acts naturally on E^N by symmetries of the potential function, then an obvious question is the existence of invariant fixed points of the semigroup. More precisely, we will say that a function $u : E^N \to \mathbb{R}$ is *invariant* if $u(r_1, \ldots, r_N) = u(Ar_1 + r, \ldots, Ar_N + r)$ for all $x = (r_1, \ldots, r_N) \in E^N$, $r \in E$ and $A \in O(E)$.

THEOREM 3. (Invariant weak KAM) *There exists an invariant and dominated function $u : E^N \to \mathbb{R}$ such that $u = T_t^- u$ for all $t \geq 0$.*

In §3, we prove the weak KAM theorem, and we study the relationship with the Hamilton–Jacobi equation. More precisely, we show that weak KAM solutions are global viscosity solutions in Ω.

An important difference in the compact case is that here the Aubry set is empty. In particular, the technique used in [**13**] to prove the invariance of all solutions is not available. Moreover, we will exhibit non-invariant solutions for the Kepler problem in a plane, which is the subject of the last section.

2. *Hölder regularity of the action potential*

This section is devoted to the study of the action potential, and to giving the proofs of Theorems 1 and 2.

2.1. *Proof of Theorem 1.* Given $r \in E$ and $R > 0$, we say that a configuration $x = (r_1, \ldots, r_N) \in E^N$ is contained in the ball $B(r, R)$ when we have $\|r_i - r\|_E \leq R$ for all $i = 1, \ldots, N$. Suppose now that we have two configurations x and y such that for some $r \in E$ and some $R > 0$, both x and y are contained in $B(r, R)$. If we tried to bound $\phi(x, y, T)$ with the action of a linear path, then two problems arise. The first one is that the linear path can present collisions, in which case the action is infinite. The second one is that, even if the linear path avoids collisions, the distance between two given bodies can be arbitrarily small for both configurations, hence the action can be arbitrarily large.

Both problems are solved in the following way: fix an intermediate configuration p with sufficiently large mutual distances, and take the linear path from x to p defined on $[0, T/2]$, followed by the linear path from p to y defined on $[T/2, T]$. This path has no more than $2N(N-1)$ collisions, and we can determine the values of $t \in [0, T]$ in which these collisions happen. Thus, reparametrizing the path in such a way that in the new times of collisions the action integral converges, we obtain the following proposition, from which we can easily deduce Theorem 1.

PROPOSITION 4. *Given two configurations $x, y \in E^N$ contained in a ball $B(r, R)$, $r \in E$, $R > 0$, and given $T > 0$, there is a curve $\gamma \in \mathcal{C}(x, y, T)$ such that $\gamma(t)$ is contained in $B(r, 6NR)$ for all $t \in [0, T]$,*

$$\frac{1}{2}\int_0^T |\dot{\gamma}(t)|^2\,dt \le \alpha T^{-1} R^2 \quad \text{and} \quad \int_0^T U_\kappa(\gamma(t))\,dt \le \beta T R^{-2\kappa},$$

where α and β are positive constants that only depend on the number of bodies, the total mass and the degree of homogeneity of the potential function. In fact, we can take

$$\alpha = 640\frac{1+\kappa}{1-\kappa} M N^4 \quad \text{and} \quad \beta = 2\frac{1+\kappa}{1-\kappa} N^{(4\kappa+2)} M^2.$$

Proof. We first observe that it suffices to give the proof for a fixed value of $T > 0$: for $S > 0$. We can define $\sigma : [0, S] \to E^N$ as $\sigma(s) = \gamma(sT/S)$, and we have

$$\int_0^S |\dot{\sigma}(s)|^2\,ds = T^2 S^{-2}\int_0^S |\dot{\gamma}(sT/S)|^2\,ds = S^{-1}T\int_0^T |\dot{\gamma}(t)|^2\,dt \le 2\alpha S^{-1}R^2,$$
$$\int_0^S U_\kappa(\sigma(s))\,ds = \int_0^S U_\kappa(\gamma(sT/S))\,ds = ST^{-1}\int_0^T U_\kappa(\gamma(t))\,dt \le \beta S R^{-2\kappa}.$$

We will then give the proof for $T = 2$. Take $v \in E$ such that $\|v\|_E = 6R$, and define $p = (p_1, \ldots, p_N) \in E^N$ by

$$p_i = r + (i-1)v, \quad i = 1, \ldots, N.$$

Therefore, the configuration p is clearly contained in $B(r, 6(N-1)R)$. Notice also that the mutual distances $p_{ij} = \|p_i - p_j\|_E$ of p are greater than $6R$ and smaller than $6(N-1)R$.

Now let $x = (r_1, \ldots, r_N)$ be a configuration such that $\|r_i - r\|_E \le R$ for all $i = 1, \ldots, N$. We consider the curve $z_x : [0, 1] \to E^N$, defined by $z_x(t) = x + \psi_x(t)(p - x)$, where $\psi_x : [0, 1] \to [0, 1]$ is an increasing function, with $\psi_x(0) = 0$ and $\psi_x(1) = 1$, to be determined. Our aim is to choose the function ψ_x conveniently, in order to obtain a bound of $A(z_x)$ which does not depend on x.

Recall that if u and v are two vectors in a Euclidean space, and $v \ne 0$, then we have, for all real numbers λ,

$$\|u + \lambda v\|^2 = \left(\lambda\|v\| + \frac{\langle u, v\rangle}{\|v\|}\right)^2 + \|u\|^2 - \frac{\langle u, v\rangle^2}{\|v\|^2}.$$

As a consequence, we get

$$\|u + \lambda v\| \ge \|v\|\left|\lambda + \frac{\langle u, v\rangle}{\|v\|^2}\right|,$$

and the minimum of $\|u+\lambda v\|$ is reached for

$$\lambda = -\frac{\langle u, v\rangle}{\|v\|^2}.$$

We will use the notations $u_{ij} = r_i - r_j$ and $v_{ij} = (p_i - p_j) - (r_i - r_j)$ for $i < j$. Thus, the mutual distances of the configuration $z_x(t)$ can be written as $d_{ij}(t) = \|u_{ij} + \psi_x(t)v_{ij}\|$. Therefore, taking $\lambda = \psi_x(t)$, $u = u_{ij}$ and $v = v_{ij}$ in the above considerations, we deduce that each mutual distance $d_{ij}(t)$ verifies

$$d_{ij}(t) \geq \|v_{ij}\|_E |\psi_x(t) - t_{ij}| \geq 4R|\psi_x(t) - t_{ij}|,$$

where

$$t_{ij} = -\frac{\langle u_{ij}, v_{ij}\rangle_E}{\|v_{ij}\|_E^2}.$$

Since $\|u_{ij}\|_E \leq 2R$ and $\|v_{ij}\|_E \geq 4R$ for all $i < j$, we obtain that $|t_{ij}| < 1/2$ for all $i < j$. Therefore, since the number of pairs (i, j) with $1 \leq i < j \leq N$ is bounded by $N^2/2$, assuming Lemma 5 below, we can choose the function ψ_x in such a way that, on the one hand,

$$\int_0^1 \dot\psi_x(t)^2\, dt \leq 5N^2 \frac{1+\kappa}{1-\kappa},$$

and, on the other hand, for each $i < j$ there is a real number s_{ij} for which

$$|\psi_x(t) - t_{ij}| \geq N^{-2}|t - s_{ij}|^{1/(1+\kappa)}$$

for all $t \in [0, 1]$. Let us estimate the action $A(z_x)$ for this function ψ_x. We have $\dot z_x(t) = \dot\psi_x(t)(p - x)$, and $\|p_i - r_i\|_E \leq 8NR$, for all $i = 1, \ldots, N$. Hence, by the previous estimates we deduce that

$$\begin{aligned}\frac{1}{2}\int_0^1 |\dot z_x(t)|^2\, dt &= \frac{1}{2}\sum_{i=1}^N m_i \|p_i - r_i\|_E^2 \int_0^1 \dot\psi_x(t)^2\, dt \\ &\leq 160\frac{1+\kappa}{1-\kappa} M N^4 R^2,\end{aligned}$$

and that

$$\begin{aligned}\int_0^1 U_\kappa(z_x(t))\, dt &= \sum_{i<j}\int_0^1 m_i m_j\, d_{ij}(t)^{-2\kappa}\, dt \\ &\leq \sum_{i<j}\int_0^1 m_i m_j (4R)^{-2\kappa} N^{4\kappa} |t - s_{ij}|^{-2\kappa/(1+\kappa)}\, dt.\end{aligned}$$

Using the fact that for $r \in (0, 1)$, and for any $s \in \mathbb{R}$, we have

$$\int_0^1 \frac{1}{|t-s|^r}\, dt = \int_{-s}^{1-s} \frac{1}{|u|^r}\, du \leq 2\int_0^1 \frac{1}{u^r}\, du = 2(1-r)^{-1},$$

we deduce that

$$\int_0^1 U_\kappa(z_x(t))\, dt \leq 2\frac{1+\kappa}{1-\kappa} M^2 N^{(4\kappa+2)} R^{-2\kappa}.$$

To finish the proof, let $y = (s_1, \ldots, s_N)$ be a second configuration contained in $B(r, R)$, and define $\gamma \in \mathcal{C}(x, y, 2)$ as follows: $\gamma(t) = z_x(t)$ if $t \leq 1$, and $\gamma(t) = z_y(2 - t)$ if $t \geq 1$. We conclude that

$$A(\gamma) = A(z_x) + A(z_y) \leq 320\frac{1+\kappa}{1-\kappa} MN^4 R^2 + 4\frac{1+\kappa}{1-\kappa} N^{(4\kappa+2)} M^2 R^{-2\kappa}.$$

This also proves the proposition for $T = 2$, with

$$\alpha = 640\frac{1+\kappa}{1-\kappa} MN^4 \quad \text{and} \quad \beta = 2\frac{1+\kappa}{1-\kappa} N^{(4\kappa+2)} M^2.$$

It is important to note that for the Newtonian case ($\kappa = 1/2$), the dependence of both constants on the number of bodies is in N^4. □

We have used the following lemma.

LEMMA 5. *Given $\kappa \in (0, 1)$ and real numbers $a_1 < \cdots < a_m$, there are real numbers $b_1 < \cdots < b_m$ and an increasing absolutely continuous homeomorphism F of* $[0, 1]$ *such that:*

(1) $$|F(t) - a_i| \geq \frac{1}{2m} |t - b_i|^{1/(1+\kappa)}$$

for all $t \in [0, 1]$ and each $i = 1, \ldots, m$; and

(2) $$\int_0^1 F'(t)^2 \, dt \leq (4 + 2a)(m + 1)\frac{1+\kappa}{1-\kappa},$$

where $a = \min\{|a_1|, \ldots, |a_m|\}$.

Proof. Given $c > 0$, let $g_c : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, defined by $g_c(x) = c|x|^{-\kappa/(1+\kappa)}$. Given $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$ such that $b_1 < \cdots < b_m$, we also define the function $f_{b,c} : \mathbb{R}\backslash\{b_1, \ldots, b_m\} \to \mathbb{R}$ by

$$f_{b,c}(t) = \max\{g_c(t - b_1), \ldots, g_c(t - b_m)\}.$$

We will define the required function F as a primitive of a function $f_{b,c}$ for a good choice of b and c. More precisely, we define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(t) = \int_0^t f_{b,c}(s) \, ds.$$

The map F is an increasing homeomorphism of $\mathbb{R}$. As for any primitive, F is absolutely continuous (see Figures 1 and 2). Moreover, we have

$$|F(t) - F(b_i)| \geq c(1 + \kappa)|t - b_i|^{1/(1+\kappa)}$$

for all $t \in \mathbb{R}$ and each $i = 1, \ldots, m$. Therefore, we must choose $c > 0$ and $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$ such that $F(1) = 1$ and $F(b_i) = a_i$ for all $i = 1, \ldots, m$.

If we fix $c > 0$, then we have a unique possible choice for b. To see this, first observe that the $m - 1$ distances between the consecutive values of a_i determine the $m - 1$ distances between the consecutive values of b_i. If we set $A_i = a_{i+1} - a_i$, and $B_i = b_{i+1} - b_i$, then we must have

$$A_i = \int_{b_i}^{b_{i+1}} f_{b,c}(s) \, ds = 2c \int_0^{B_i/2} s^{-\kappa/(1+\kappa)} \, ds = 2^{\kappa/(1+\kappa)} c(1 + \kappa) B_i^{1/(1+\kappa)},$$

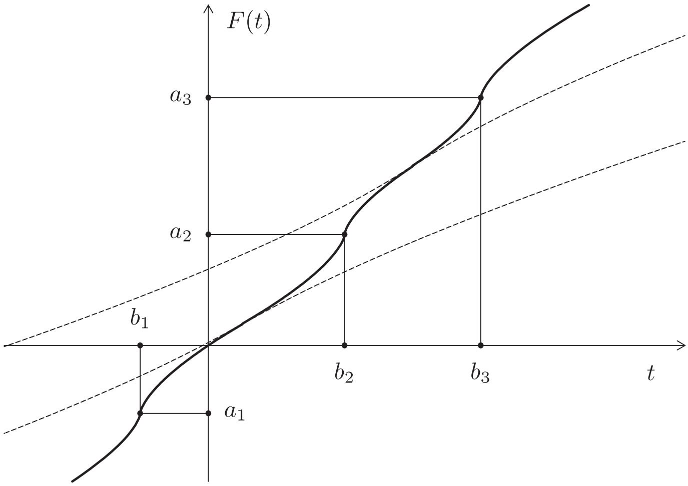

FIGURE 1. Graph of $F(t) = \int_0^t f_{b,c}(s)\, ds$.

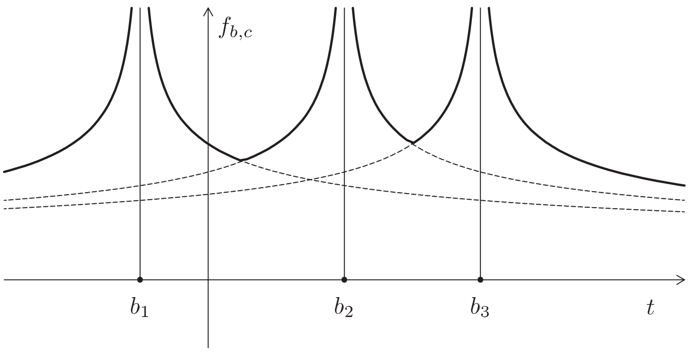

FIGURE 2. Graph of $f_{b,c}$ for $b = (b_1, b_2, b_3)$.

hence, $B_i = 2^{-\kappa}[A_i/c(1+\kappa)]^{1+\kappa}$. From the condition $F(b_1) = a_1$, we deduce that

$$a_1 = \int_0^{b_1} f_{b,c}(s)\, ds = -\int_0^{-b_1} f_{b,c}(s + b_1)\, ds.$$

Therefore, b_1 must be the unique solution of the equation $\int_0^{-x} f_{b',c}(s)\, ds = -a_1$, where

$$b' = (0, B_1, B_1 + B_2, \ldots, B_1 + \cdots + B_{m-1}) = (0, b_2 - b_1, \ldots, b_m - b_1).$$

Moreover, we have shown that there is a continuous vector $b(c) \in \mathbb{R}^m$ such that $\int_0^{b_i} f_{b(c),c}(s)\, ds = a_i$ for all $i = 1, \ldots, m$. It is therefore clear that

$$\delta(c) = \int_0^1 f_{b(c),c}(s)\, ds$$

also depends continuously on c. We claim that $c \in [1/2m(1+\kappa), 2+a]$ exists for which $\delta(c) = 1$. We have

$$\delta(c) = \int_0^1 f_{b(c),c}(s)\, ds \le \sum_{i=1}^m \int_0^1 g_c(s - b_i(c))\, ds.$$

Since

$$\int_0^1 g_c(s - b_i(c))\, ds = \int_{-b_i(c)}^{1-b_i(c)} c|u|^{-\kappa/(1+\kappa)}\, du \leq 2c \int_0^1 u^{-\kappa/(1+\kappa)}\, du = 2c(1+\kappa),$$

we deduce that $\delta(c) < 1$ when $c < 1/2m(1+\kappa)$. In order to prove the claim, it suffices to show that $\delta(c) > 1$ when $c > 2 + a$. Since $a = |a_j|$ for some $j \in \{1, \ldots, m\}$, we have

$$\begin{aligned} a &= \left| \int_0^{b_j(c)} f_{b(c),c}(s)\, ds \right| \\ &\geq \left| \int_0^{b_j(c)} c|s - b_j(c)|^{-\kappa/(1+\kappa)}\, ds \right| \\ &\geq c \int_0^{|b_j(c)|} s^{-\kappa/(1+\kappa)}\, ds = c(1+\kappa)|b_j(c)|^{1/(1+\kappa)}, \end{aligned}$$

which implies $|b_j(c)| \leq [a/c(1+\kappa)]^{(1+\kappa)}$. On the other hand, we have

$$\begin{aligned} \delta(c) &\geq \min\{g_c(s - b_j(c)) \mid s \in [0, 1]\} \\ &\geq c(1 + |b_j(c)|)^{-\kappa/(1+\kappa)}. \end{aligned}$$

Thus, it suffices to prove that $|b_j(c)| \leq c^{(1+\kappa)/\kappa} - 1$ when $c > 2 + a$. By the previous estimation of $|b_j(c)|$, we only have to prove that $(a/1+\kappa)^{1+\kappa} \leq c^{1+\kappa}(c^{(1+\kappa)/\kappa} - 1)$, but this condition is clearly satisfied if $c > 2$ and $c > a$.

We take $c \in [1/2m(1+\kappa), 2 + a]$ such that $\delta(c) = 1$ and we define $F : [0, 1] \to [0, 1]$ by

$$F(t) = \int_0^t f_{b(c),c}(s)\, ds.$$

In order to see that this function satisfies all the required conditions, it remains to estimate the L^2 norm of F'. If we observe that $[0, 1] \backslash \{b_1(c), \ldots, b_m(c)\}$ has at most $m + 1$ components I_j, and that on each one of these components we have

$$\int_{I_j} f_{b(c),c}(s)^2\, ds \leq 2 \int_0^1 cs^{-2\kappa/(1+\kappa)}\, ds,$$

we conclude that

$$\int_0^1 F'(t)^2\, dt \leq (4 + 2a)(m + 1)\frac{1+\kappa}{1-\kappa}. \qquad \square$$

2.2. *Minimal configurations.* The following observations show that Theorem 1 is optimal in the sense that the bound is reached by some configurations. We will first recall the notions of *central* and *minimal* configurations, as well as some properties (see for instance Wintner [**16**], where the Newtonian case is discussed).

We say that a configuration $x \in E^N$ is *minimal* if it is a minimum of the potential function U_κ restricted to the sphere $\{y \in E^N \mid I(y) = I(x)\}$. In particular, minimal configurations are *central* configurations, that is critical points of $\widetilde{U}_\kappa = I^\kappa U_\kappa$, or, in other words, configurations $x \in E^N$ which are critical points of U_κ restricted to $\{y \in E^N \mid I(y) = I(x)\}$. Central configurations are also characterized as configurations which admit homothetic motions. In other words, a configuration $x_0 \in E^N$ is central if and only if $U_\kappa(x_0) < +\infty$ and $x(t) = r(t)x_0$ is a solution of the N-body problem for some positive real function $r(t)$.

Take $x_0 \in E^N$, a central configuration. If we look for a homothetic motion through x_0, then we must solve a one-dimensional differential equation satisfied by $r(t)$. A particular solution, which we will call parabolic, is given by $x(t) = ct^{1/(1+\kappa)}x_0$ for some value of $c > 0$. A simple computation shows that the action of this solution is

$$\begin{aligned} A(x|_{[0,T]}) &= \frac{c^2}{2(1+\kappa)^2}\int_0^T t^{-2\kappa/(1+\kappa)}\,dt + c^{-2\kappa}U_\kappa(x_0)\int_0^T t^{-2\kappa/(1+\kappa)}\,dt \\ &= \left(\frac{c^2}{2(1-\kappa^2)} + c^{-2\kappa}U_\kappa(x_0)\frac{1+\kappa}{1-\kappa}\right)T^{(1-\kappa)/(1+\kappa)}. \end{aligned}$$

If we set $R_T = \|x(T)\| = T^{1/(1+\kappa)}\|cx_0\|$, then we can write

$$x|_{[0,T]}) = \alpha_0 T^{-1}R_T^2 + \beta_0 T R_T^{-2\kappa}$$

for a good choice of constants α_0 and β_0.

On the other hand, we will prove that if x_0 is a minimal configuration, then the above solution $x(t)$ is globally minimizing. In other words, we have

$$\phi(0, x(T), T) = A(x|_{[0,T]}),$$

for all $T > 0$, therefore the bound for $\phi(x, y, T)$ given by Theorem 1 cannot be improved modulo the choice of the constants. We will assume for simplicity that x_0 is also normal, meaning that $I(x_0) = 1$.

In order to prove that x is globally minimizing, we will first study the homogeneous one-center problem in dimension one which is satisfied by the function $r(t) = I(x(t))^{1/2}$. More precisely, we have that $r(t)$ must be an extremal for the Lagrangian system in $\mathbb{R}^+ = [0, +\infty)$ defined by

$$L_0(r, v) = \frac{v^2}{2} + \frac{U_0}{r^{2\kappa}},$$

where $U_0 = U\kappa(x_0)$.

It can easily be proved using the lower semiconinuity of the Lagrangian action of L_0 that, given $0 \le r_1 < r_2$, there is at least one absolute minimizer between r_1 and r_2. This means that there is a curve $\gamma(r_1, r_2) : [0, T] \to \mathbb{R}^+$, for some positive time $T > 0$, such that $\gamma(r_1, r_2)(0) = r_1$, $\gamma(r_1, r_2)(T) = r_2$, and such that $\gamma(r_1, r_2)$ minimizes the Lagrangian action in the set of all absolutely continuous curves $\sigma : [a, b] \to \mathbb{R}^+$ with $\sigma(a) = r_1$, $\sigma(b) = r_2$ and $a < b$. Moreover, the fact that this absolute minimization property holds (i.e. with fixed extremities but in free time), implies that the energy of the extremal $\gamma(r_1, r_2)$ must be critical (zero). Therefore, $\gamma(r_1, r_2)$ satisfies the differential equation $\dot\gamma^2 = 2U_0\gamma^{-2\kappa}$ and we conclude the uniqueness of such an absolute minimizer. By integration, we get that for any $r > 0$, the absolute minimizer $\gamma(0, r) : [0, T] \to \mathbb{R}^+$ is defined for $T = (1+\kappa)(2U_0)^{-1/2}r^{(1+\kappa)}$ and that $\gamma(0, r)(t) = ct^{1/(1+\kappa)}$, where $c = (2U_0)^{1/(2+2\kappa)}(1+\kappa)^{1/(1+\kappa)}$.

We now prove now that if x_0 is a minimal configuration, then the parabolic motion $x(t) = ct^{1/(1+\kappa)}x_0$ defined above is globally minimizing. Fix $T > 0$, and take any other curve $\gamma \in \mathcal{C}(0, x(T), T)$. We must to prove that $A(\gamma) \ge A(x|_{[0,T]})$. In fact, we will prove that if $S = \sup\{t \in [0, T] \mid \gamma(t) = 0\}$, then

$$A(\gamma|_{[S,T]}) \ge A(x|_{[0,T]}).$$

Setting $\gamma_1 = \gamma|_{[S,T]}$, we have $\gamma_1(t) \neq 0$ for all $t \in (S, T]$. Thus, we can set $\gamma_1(t) = r(t)s(t)$, where $r(t) = |\gamma_1(t)| = I(\gamma_1(t))^{1/2}$ for $t > S$. Obviously, we have that $|s(t)| = 1$ for all $t > S$, and the action of γ_1 can be written as

$$A(\gamma_1) = \frac{1}{2}\int_S^T \dot{r}(t)^2\,dt + \frac{1}{2}\int_S^T r(t)^2|\dot{s}(t)|^2\,dt + \int_S^T r(t)^{-2\kappa}U_\kappa(s(t))\,dt.$$

Since x_0 is minimal, we have $U(s(t)) \geq U_0 = U\kappa(x_0)$ for all $t > S$. Moreover, we have

$$A(\gamma_1) \geq A(rx_0) = \int_S^T \left(\frac{1}{2}\dot{r}(t)^2\,dt + U_0 r(t)^{-2\kappa}\right)dt.$$

Note that the last integral is simply the Lagrangian action of the curve $r(t) : [S, T] \to \mathbb{R}^+$ for the one-dimensional homogeneous one-center problem, and, as we have see the absolute minima of the Lagrangian action, is reached by the zero-energy solution $r(t) = ct^{1/(1+\kappa)}$. However, in that case, the last integral is also a Lagrangian action of the parabolic motion $x(t)$ restricted to the interval $[0, T]$ for the N-body problem. We deduce that

$$A(\gamma) \geq A(\gamma_1) \geq \int_0^T \left(\frac{1}{2}\dot{r}(t)^2\,dt + U_0 r(t)^{-2\kappa}\right)dt = A(x|_{[0,T]}),$$

hence, we conclude that the solution $x(t)$ is globally minimizing.

2.3. *Properties of the action potential and proof of Theorem 2.* We start this section by showing that the action potential is a distance function on E^N.

PROPOSITION 6. *For all $x, y \in E^N$, we have $\phi(x, y) = 0$ if and only if $x = y$.*

Proof. Let $x \in E^N$ be a configuration, and choose a path $\sigma : [0, 1] \to E^N$ which satisfies $\sigma(0) = x$ and $A(\sigma) < +\infty$. We then define for $0 < T \leq 2$ the curve $\gamma_T \in \mathcal{C}(x, x, T)$ by $\gamma_T(t) = \sigma(t)$ if $t \leq T/2$, and $\gamma_T(t) = \sigma(T - t)$ if $t \geq T/2$. It is not difficult to see that $A(\gamma_T) \to 0$ as $T \to 0$, from which it follows that $\phi(x, x) = 0$ for all $x \in E^N$.

To see that the condition is necessary, take any two configurations $x = (r_1, \ldots, r_N)$ and $y = (s_1, \ldots, s_N)$ in E^N, and a path $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathcal{C}(x, y)$. If $d = \|y - x\|$, and γ is defined on $[0, T]$, then $T_0 \in [0, T]$ must exist such that $\|\gamma(T_0) - x\| = d$ and $\|\gamma(t) - x\| \leq d$ for all $t \in [0, T_0]$. Moreover, we must have $d = \|\gamma_i(T_0) - r_i\|_E$ for some $i \in \{1, \ldots, N\}$. If $T_0 \geq 1$, we can write

$$A(\gamma) \geq A(\gamma|_{[0,T_0]}) \geq \int_0^{T_0} U_\kappa(\gamma(t))\,dt \geq C > 0,$$

where $C = \min\{U_\kappa(z) \mid \|z - x\| \leq d\}$. If $T_0 \leq 1$, we have

$$A(\gamma) \geq A(\gamma|_{[0,T_0]}) \geq \frac{m_i}{2}\int_0^{T_0} \|\dot{\gamma}_i(t)\|_E^2\,dt \geq \frac{md^2}{2},$$

where $m = \min\{m_1, \ldots, m_N\}$. The last inequality follows from the fact that γ_i is absolutely continuous and from the Cauchy–Schwartz inequality. Therefore, we conclude that if $\phi(x, y) = 0$, then $d = 0$ and $x = y$. □

In the following, we will denote by $\delta(z)$ the minimal distance between the bodies of the configuration z. More precisely, $\delta : E^N \to \mathbb{R}^+$ will be the function defined by $\delta(z) = \min\{\|z_i - z_j\|_E \mid i < j\}$, where $z = (z_1, \ldots, z_N)$. Thus, the set of configurations without collisions is simply $\Omega = \{z \in E^N \mid \delta(z) > 0\}$. From the following proposition, we can easily deduce that the action potential is a locally Lipschitz function in $\Omega \times \Omega$.

PROPOSITION 7. *There are continuous functions $k(z) > 0$ and $\epsilon(z) > 0$ in Ω such that, if $x, z \in E^N$ satisfy $\|x\| < \epsilon(z)$, then $\phi(z, z + x) \leq k(z)\|x\|$.*

Proof. We give the proof for $\epsilon(z) = \delta(z)/4$. Since z is without collisions, we have $\epsilon(z) > 0$. For $T > 0$, we define the curve $\gamma : [0, T] \to E^N$ by $\gamma(t) = z + (t/T)x$. If $z = (z_1, \ldots, z_N)$ and $x = (r_1, \ldots, r_N)$, then $\gamma(t) = (\gamma_1(t), \ldots, \gamma_N(t))$, where $\gamma_i(t) = z_i + (t/T)r_i$. Hence, for $i < j$ and $t \in [0, T]$, we can write

$$\gamma_{ij}(t) = \|\gamma_i(t) - \gamma_j(t)\|_E \geq \|z_i - z_j\|_E - (t/T)\|r_i - r_j\|_E \geq \delta(z)/2,$$

and

$$U_\kappa(\gamma(t)) = \sum_{i<j} m_i m_j \gamma_{ij}(t)^{-2\kappa} \leq M^2 N^2 [\delta(z)/2]^{-2\kappa}.$$

Therefore, using the fact that $|x|^2 = I(x) \leq MN\|x\|^2$, we deduce that

$$\begin{aligned} A(\gamma) &= \frac{1}{2}\int_0^T |x/T|^2\, dt + \int_0^T U(\gamma(t))\, dt \\ &\leq MN\|x\|^2/2T + M^2N^2[\delta(z)/2]^{-2\kappa}T. \end{aligned}$$

If $x = 0$, there is nothing to prove, since we already know that $\phi(z, z) = 0$. If $x \neq 0$, we can take $T = \|x\|$, and the above estimation gives $A(\gamma) \leq k(z)\|x\|$ for $k(z) = MN/2 + M^2N^2(\delta(z)/2)^{-2\kappa}$. □

We now introduce the notion of *cluster partition* of a subset $A \subset E$, adapted to our purposes. Given $\lambda > 1$, we will say that the set $\{r_1, \ldots, r_K\} \subset E$ defines a λ-cluster partition of size $R > 0$ of A if the following two conditions are satisfied:

(1) $\|r_i - r_j\|_E \geq 2\lambda R$ for all $1 \leq i < j \leq K$;

(2) A is contained in the union $\bigcup_{i=1}^K B(r_i, R)$.

It is clear that if A is finite and R is small enough, then A itself defines a cluster partition of size R of A. It is also clear that if A is bounded, then any $r \in A$ defines a trivial cluster partition of size R for any $R > \mathrm{diam}(A)$.

We will need the following lemma.

LEMMA 8. *Given $\lambda > 1$, $A = \{r_1, \ldots, r_N\} \subset E$ and $\epsilon > 0$, there is a subset $A' \subset A$, and $R(\epsilon) > 0$, such that:*

(i) *$\epsilon \leq R(\epsilon) < (2\lambda)^N \epsilon$;*

(ii) *A' defines a λ-cluster partition of size $R(\epsilon)$ of A.*

Proof. We reason recursively. We begin by setting $A'_1 = A$. If A'_1 does not define a λ-cluster partition of size ϵ, then there are $r, s \in A'_1$ such that $\|r - s\|_E < 2\lambda\epsilon$. If that is the case, we define $A'_2 = A'_1 \backslash \{s\}$. We then reason as before: if A'_2 does not define a λ-cluster partition of size $2\lambda\epsilon$, then we have $r, s \in A'_2$ such that $\|r - s\|_E < (2\lambda)^2\epsilon$, and we set $A'_3 = A'_2 \backslash \{s\}$. It is clear that the process finishes in at most N steps. □

Using the existence of cluster partitions, we will prove the following proposition, from which Theorem 2 can be deduced as a simple corollary.

PROPOSITION 9. *There are positive constants $\alpha_1, \beta_1 > 0$ such that, for all $x, y \in E^N$ and for all $T > 0$, we have*

$$\phi(x, y, T) \leq \alpha_1 T^{-1}\epsilon^2 + \beta_1 T\epsilon^{-2\kappa},$$

whenever $\epsilon > \|x - y\|$. The constants α_1 and β_1 only depend on the degree of homogeneity of the potential (-2κ), the number of bodies N and their masses.

Proof. Fix a configuration $x = (r_1, \ldots, r_N) \in E^N$, and denote by A_x the set $\{r_1, \ldots, r_N\} \subset E$. Let $y = (s_1, \ldots, s_N) \in E^N$ be any other configuration. If we apply Lemma 8 to A_x with $\epsilon > \|y - x\|$ and $\lambda = 24N$, we conclude that there are $r_{i_1}, \ldots, r_{i_K} \in A_x$, and $R(\epsilon) > 0$, with the following properties:

(1) $\epsilon \leq R(\epsilon) < (48N)^N\epsilon$;

(2) for all $1 \leq j < k \leq K$, we have $\|r_{i_j} - r_{i_k}\|_E \geq 48NR(\epsilon)$; and

(3) $A_x \cup A_y$ is contained in the disjoint union $\bigcup_{j=1}^{K} B_j$, where $B_j = B(r_{i_j}, 2R(\epsilon))$.

Therefore, both configurations x and y are decomposed in K clusters, each one contained in a ball B_j. More precisely, we have a partition $\{1, \ldots, N\} = I_1 \cup \cdots \cup I_K$ such that $i \in I_j$ if and only if both r_i and s_i are in B_j. Denote by $N_j = \mathrm{card}(I_j)$ the number of bodies in cluster j, and by M_j the total mass of this cluster, that is $M_j = \sum_{i\in I_j} m_i$. Thus, we have $N = N_1 + \cdots + N_K$ and $M = M_1 + \cdots + M_K$.

We consider now the N_j-body problem composed of the bodies in the ball B_j. Given $T > 0$, we apply Proposition 4 in each ball B_j, $j = 1, \ldots, K$, with initial and final conditions conformed by the N_j bodies of x and y contained in B_j. We therefore obtain a path $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathcal{C}(x, y, T)$ such that for all $j = 1, \ldots, K$ we have:

(1) if $i \in I_j$, then $\gamma_i(t) \in B(r_{i_j}, 12NR(\epsilon))$ for all $t \in [0, T]$;

(2)
$$T_j = \frac{1}{2}\int_0^T \sum_{i\in I_j} m_i\|\dot\gamma_i(t)\|_E^2\,dt \leq 10^6\frac{1+\kappa}{1-\kappa}M_jN_j^6R(\epsilon)^2/T;$$

and

(3)
$$W_j = \int_0^T \sum_{i,k\in I_j}^{i<k} m_im_k\|\gamma_i(t) - \gamma_k(t)\|_E^{-2\kappa}\,dt$$
$$\leq 2\frac{1+\kappa}{1-\kappa}N_j^{2\kappa+2}M_j^2 12^{-2\kappa}R(\epsilon)^{-2\kappa}T.$$

Notice that the action of the curve $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathcal{C}(x, y, T)$ is

$$A(\gamma) = \sum_{j=1}^{K} T_j + \sum_{j=1}^{K} W_j + W_0,$$

where W_0 is the integral of the terms of the potential function U_κ corresponding to pairs of bodies in different clusters. More precisely,

$$W_0 = \int_0^T \sum_{1\leq j<l\leq K}\ \sum_{i\in I_j, k\in I_l} m_im_k\|\gamma_i(t) - \gamma_k(t)\|_E^{-2\kappa}\,dt.$$

Since the balls $B(r_{i_j}, 24NR(\epsilon))$ are disjoint, we deduce that

$$W_0 \leq N^2 M^2 (24N)^{-2\kappa} R(\epsilon)^{-2\kappa} T.$$

Using the fact that $R(\epsilon) < (24N)^N \epsilon$, we can write

$$A(\gamma) < \alpha_1 T^{-1} \epsilon^2 + \beta_1 T \epsilon^{-2\kappa}$$

for some positive constants α_1 and β_1 only depending on N, M and κ. □

COROLLARY 10. *There is a positive constant $\mu > 0$ such that for all $x \in E^N$*

$$\phi(x, x, T) \leq \mu T^{(1-\kappa)/(1+\kappa)}.$$

Proof. It suffices to take $\epsilon = T^{1/(1+\kappa)}$ in Proposition 9. □

Proof of Theorem 2. By the previous corollary, we know that $\phi(x, x) = 0$ for all $x \in E^N$. If $x, y \in E^N$ are two different configurations, then Proposition 9 says that

$$\phi(x, y, T) < \alpha_1 T^{-1} \epsilon^2 + \beta_1 T \epsilon^{-2\kappa}$$

for all $T > 0$ and for any $\epsilon > \|x - y\|$. Since the right-hand side of this inequality is a continous function of $\epsilon > 0$, we also have

$$\phi(x, y, T) \leq \alpha_1 T^{-1} \|x - y\|^2 + \beta_1 T \|x - y\|^{-2\kappa},$$

and the proof is achieved taking $T = \|x - y\|^{1+\kappa}$. □

2.4. *Homogeneity of the action potential.* There is a property of homogeneity of the action potential resulting from the homogeneity of the potential function U_κ. We did not use this property in the above proofs, but we think that it is useful to complete the picture of the action potential. The proof can be done by reparametrizing conveniently homothetic paths of a given path.

PROPOSITION 11. *If $\lambda > 0$, then $\phi(\lambda x, \lambda y) = \lambda^{1-\kappa} \phi(x, y)$ for all $x, y \in E^N$.*

3. *Weak KAM theory*

The relationship between global solutions of the Hamilton–Jacobi equation and globally minimizing solutions of the corresponding Lagrangian flow is well known. Let us recall that the *Hamiltonian*, defined on $T^*E^N = E^N \times (E^*)^N$, is the function

$$H(x, p) = \tfrac{1}{2}|p|^2 - U_\kappa(x),$$

where $|p|$ denotes the dual norm of $p \in (E^*)^N$ with respect to the norm on E^N induced by the mass scalar product. More precisely, if we identify the space E with its dual E^* using the scalar product $\langle , \rangle_E$, and $p = (p_1, \ldots, p_N) \in (E^*)^N$, then

$$|p|^2 = \sum_{i=1}^{N} m_i^{-1} \|p_i\|_E^2.$$

A closely related function is the *total energy*, defined on TE^N as $\mathcal{E} = H \circ \mathcal{L}$, where $\mathcal{L} : TE^N \to T^*E^N$ is the Legendre transform $\mathcal{L}(x; v_1, \ldots, v_N) = (x; p_1, \ldots, p_N)$, $p_i = m_i v_i$. It is easy to see that $\mathcal{E}$ is a first integral of the motion.

We will prove the existence of critical global (weak) solutions for the Hamilton–Jacobi equation $H(x, d_x u) = c$. The critical value of this Hamiltonian can be defined as the infimum of the values of $c \in \mathbb{R}$ such that the Hamilton–Jacobi equation admits global subsolutions. Since $\inf_{E^N} U_\kappa(x) = 0$, and constant functions are global subsolutions for $c = 0$, it follows that the critical value is $c = 0$. Therefore, we are interested in global solutions of

$$|d_x u|^2 = 2U_\kappa(x). \tag{HJ}$$

We will obtain global solutions as fixed points of a continuous semigroup acting on the set of weak subsolutions, namely the Lax–Oleinik semigroup. There are no new ideas in the method that we apply here. In fact, we will follow the scheme introduced by Fathi in [**10**], with some adjustments to our setting. As we have said in the introduction, the difference is that we consider a space of Hölder functions on which the semigroup acts, and Theorem 2 will assure that the method works with this space.

3.1. *The Lax–Oleinik semigroup.* Given a continuous function $u : E^N \to \mathbb{R}$ and $t > 0$, we define $T_t^- u : E^N \to [-\infty, +\infty)$ by

$$T_t^- u(x) = \inf\{u(y) + \phi(x, y, t) \mid y \in E^N\}.$$

We also define $T_0^- u = u$ for all functions u. The semigroup property follows from the definition. In other words, for any function u we have that $T_t^-(T_s^- u) = T_{t+s}^- u$ for all $t, s \geq 0$. We will restrict the semigroup to the set $\mathcal{H}$ of dominated functions. More precisely, we define

$$\mathcal{H} = \{u : E^N \to \mathbb{R} \mid u(x) - u(y) \leq \phi(x, y) \text{ for all } x, y \in E^N\}.$$

Notice that $u : E^N \to \mathbb{R}$ is in $\mathcal{H}$ if and only if $u \leq T_t^- u$ for all $t \geq 0$. On the other hand, $u \leq v$ implies that $T_t^- u \leq T_t^- v$ for all $t \geq 0$. Therefore, the semigroup property implies that $T_t^- u \in \mathcal{H}$ for all $u \in \mathcal{H}$. Also notice that $\mathcal{H}$ is convex and non-empty since it contains all constant functions.

In the following, the set $\mathcal{H}$ will be endowed with a compact open topology, that is the topology generated by the sets

$$U_K(u, \epsilon) = \{v \in \mathcal{H} \mid |v(x) - u(x)| < \epsilon \text{ for all } x \in K\},$$

with $u \in \mathcal{H}$, $K \subset E^N$ compact, and $\epsilon > 0$.

PROPOSITION 12. *The map $T^- : \mathcal{H} \times [0, +\infty) \to \mathcal{H}$, $(u, t) \mapsto T_t^- u$ is continuous.*

We will use the following lemma.

LEMMA 13. *For all $x, y \in E^N$ and $T > 0$, we have $\phi(x, y, T) \geq (m/2T)\|x - y\|^2$, where $m = \min\{m_1, \ldots, m_N\}$.*

Proof. Let $r, s \in E$ and $\sigma : [0, T] \to E$ be an absolutely continuous curve such that $\sigma(0) = r$ and $\sigma(T) = s$. We observe that

$$\|r - s\|_E \leq \int_0^T \|\dot\sigma(t)\|_E \, dt \leq T^{1/2}\left(\int_0^T \|\dot\sigma(t)\|_E^2 \, dt\right)^{1/2},$$

hence,

$$\|r - s\|_E^2 \le T \int_0^T \|\dot\sigma(t)\|_E^2 \, dt.$$

If $x = (r_1, \ldots, r_N)$ and $y = (s_1, \ldots, s_N)$ are two configurations, then we can choose $i \in \{1, \ldots, N\}$ such that $\|r_i - s_i\|_E = \|x - y\|$. Now take $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathcal{C}(x, y, T)$. By the previous observation, we have

$$A(\gamma) \ge (m_i/2) \int_0^T \|\dot\gamma_i(t)\|_E^2 \, dt \ge (m_i/2T)\|r_i - s_i\|_E^2 \ge (m/2T)\|x - y\|^2,$$

which proves the lemma since $\phi(x, y, T) = \inf\{A(\gamma) \mid \gamma \in \mathcal{C}(x, y, T)\}$. □

Proof of Proposition 12. As a first step, we show that given $R > 0$ and $t > 0$, there is a constant $k(R, t) > 0$ such that

$$T_t^- u(x) = \inf\{u(y) + \phi(x, y, t) \mid \|y - x\| \le k(R, t)\}$$

for all $u \in \mathcal{H}$ and all $x \in E^N$ with $\|x\| \le R$. To see this, fix $R > 0$, $t > 0$, $u \in \mathcal{H}$ and $x \in E^N$ such that $\|x\| \le R$. Suppose that $y \in E^N$ is such that $\|y - x\| > 1$ and $u(y) + \phi(x, y, t) \le u(x) + \phi(x, x, t)$. Then, by Lemma 13 and Theorem 2, we have

$$\frac{m}{2t}\|y - x\|^2 \le \eta\|y - x\|^{1-\kappa} + \phi(x, x, t).$$

Therefore, using the fact that $\|y - x\| > 1$ and Theorem 1 we deduce

$$m\|y - x\|^2 \le 2\eta t\|y - x\| + 2\alpha R^2 + 2\beta t^2 R^{-2\kappa},$$

and hence that $\|y - x\| \le k_0(R, t)$, where

$$k_0(R, t) = \eta t/m + (\eta^2 t^2/m^2 + 2\alpha R^2/m + 2\beta t^2 R^{-2\kappa}/m)^{1/2}.$$

Setting $k(R, t) = \max\{1, k_0(R, t)\}$, it follows that $u(y) + \phi(x, y, t) > u(x) + \phi(x, x, t)$ for all $y \in E^N$ such that $\|y - x\| > k(R, t)$, and we conclude that

$$\begin{aligned} T_t^- u(x) &= \inf\{u(y) + \phi(x, y, t) \mid y \in E^N\} \\ &= \inf\{u(y) + \phi(x, y, t) \mid \|y - x\| \le k(R, t)\}. \end{aligned}$$

Now let $u, v \in \mathcal{H}$ and $t > 0$. Let $K \subset E^N$ be a compact subset, and $R > 0$ such that $\|x\| \le R$ for all $x \in K$. If we set

$$K_t = \bigcup_{x \in K} \{y \in E^N \mid \|y - x\| \le k(R, t)\},$$

then for all $x \in K$ we have $T_t^- v(x) = \inf\{v(y) + \phi(x, y, t) \mid y \in K_t\}$. On the other hand, since

$$v(y) \le u(y) + \sup\{|u(y) - v(y)| \mid y \in K_t\} \quad \text{for all } y \in K_t,$$

we deduce that

$$T_t^- v(x) \le \inf\{u(y) + \phi(x, y, t) \mid y \in K_t\} + \sup\{|u(y) - v(y)| \mid y \in K_t\}.$$

Thus, we have proved that

$$T_t^- v(x) - T_t^- u(x) \leq \sup\{|u(y) - v(y)| \mid y \in K_t\} \quad \text{for all } x \in K.$$

Hence, we have that

$$|T_t^- v(x) - T_t^- u(x)| \leq \sup\{|u(y) - v(y)| \mid y \in K_t\}$$

for all $x \in K$. Since the subset $K_t \subset E^N$ is compact, this implies the continuity of the map T_t^- for each $t \geq 0$. It remains to prove the continuity of T_t^- with respect to t. Since $(T_t^-)_{t \geq 0}$ is a semigroup, it suffices to prove the continuity at $t = 0$. Given $u \in \mathcal{H}$, we have by Corollary 10 that

$$0 \leq T_t^- u(x) - u(x) \leq \phi(x, x, t) \leq \mu t^{(1-\kappa)/(1+\kappa)}$$

for all $x \in E^N$.

Therefore, $T_t^- u$ converges uniformly to u when $t \to 0$. As we have said, using the semigroup property, we can deduce that $T_t^- u$ converges uniformly to $T_{t_0}^- u$ when $t \to t_0$ for all $t_0 \geq 0$. □

3.2. *Proof of Theorem 3.* Let $\widehat{\mathcal{H}}$ be the quotient space of $\mathcal{H}$ by the subspace of constant functions. Thus, $\widehat{\mathcal{H}}$ is homeomorphic to $\mathcal{H}_0 = \{u \in \mathcal{H} \mid u(0) = 0\}$. By Theorem 2, we have that dominated functions are uniformly equicontinuous. It follows that $\mathcal{H}_0$ is compact, by Ascoli's theorem. Therefore, $\widehat{\mathcal{H}}$ is a compact, convex and non-empty subset of $\widehat{C^0}(E^N, \mathbb{R})$, the quotient of the vector space $C^0(E^N, \mathbb{R})$ by the subspace of constant functions. Notice that $\widehat{C^0}(M, \mathbb{R})$ is endowed with the quotient topology of the compact open topology on $C^0(M, \mathbb{R})$. In particular, $\widehat{C^0}(M, \mathbb{R})$ is a locally convex topological vector space.

Since $T_t^-(u + c) = T_t^- u + c$ for all $c \in \mathbb{R}$, it is clear that the semigroup T^- canonically defines a continuous semigroup $\widehat{T}_t^- : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$. If we apply the Schauder–Tykhonov theorem (see [**8**, pp. 414–415]), we conclude that $\widehat{T}_t^-$ has a fixed point in $\widehat{\mathcal{H}}$, that is there is a function $u \in \mathcal{H}$ such that $T_t^- u = u + c(t)$ for some function $c : [0, +\infty) \to \mathbb{R}$. The semigroup property and the continuity of T^- imply that $c(t) = c(1)t$. Since $u \in \mathcal{H}$, we have that $u \leq T_t^- u$ for all $t \geq 0$, hence we must have $c(1) \geq 0$. We will prove that $c(1) = 0$. Notice that $T_t^- u = u + c(1)t$ implies $u(x) - u(y) \leq \phi(x, y, t) - c(1)t$ for all $x, y \in E^N$. Hence, by Theorem 1, we have that

$$u(x) - u(y) \leq \alpha \frac{R^2}{t} + \left(\frac{\beta}{R^{2\kappa}} - c(1) \right) t,$$

whenever x and y are contained in a ball of E of radius $R > 0$. Since this must be true for R and t arbitrarily large, we conclude that $c(1) = 0$. Therefore, $T_t^- u = u$ for all $t \geq 0$.

It remains to prove that there are fixed points of T^- which are invariant by the group of symmetries. This can be done as in [**13**] as follows. We define $\mathcal{H}_{\text{inv}}$ as the set of functions in $\mathcal{H}$ which are invariant by symmetries. Thus $\mathcal{H}_{\text{inv}}$ is also convex, closed and non-empty since constant functions are invariant. Moreover, $\mathcal{H}_{\text{inv}}$ is stable by the Lax–Oleinik semigroup. Therefore, the quotient of this set by the subspace of constant functions is also compact, convex, non-empty and stable by the induced semigroup $\widehat{T}^-$. With the same arguments as above, we obtain an invariant fixed point. □

3.3. *Viscosity solutions and subsolutions.* It is well known that the notion of dominated functions is related to a notion of the subsolutions of the Hamilton–Jacobi equation, namely the notion of viscosity subsolutions. On the other hand, viscosity solutions (see below) can be detected as fixed points, modulo constants, of the Lax–Oleinik semigroup. An introduction to the subject of viscosity solutions can be found in the books [**1, 2**] or [**9**], for example. However, our setting presents some technical differences, essentially due to the fact that the potential function is infinite in the set of configurations with collisions. The following is a slight adaptation of some results in [**11**, §5].

Recall that $u : E^N \to \mathbb{R}$ is a *viscosity subsolution* at $x_0 \in E^N$ of (HJ), if for each C^1 function $\psi : E^N \to \mathbb{R}$ such that x_0 is a maximum of $u - \psi$, we have $|d_{x_0}\psi|^2 \leq 2U_\kappa(x_0)$. Given $V \subset E^N$, we say that u is a viscosity subsolution in V if it is a viscosity subsolution at each $x \in V$. We remark that any function is trivially a viscosity subsolution in Ω^c, where $\Omega \subset E^N$ denotes the set of configurations without collisions.

Analogously, a function $u : E^N \to \mathbb{R}$ is said to be a *viscosity supersolution* at $x_0 \in E^N$ of (HJ), if for each C^1 function $\psi : E^N \to \mathbb{R}$ such that x_0 is a minimum of $u - \psi$, we have $|d_{x_0}\psi|^2 \geq 2U_\kappa(x_0)$. If $x_0 \in \Omega^c$, then u is a viscosity supersolution at x_0 if and only if there are no C^1 functions ψ such that x_0 is a minimum of $u - \psi$. As for subsolutions, given $V \subset E^N$, we say that u is a viscosity supersolution in V if it is a viscosity supersolution at each $x \in V$.

We say that a continuous function $u : E^N \to \mathbb{R}$ is a *viscosity solution* of (HJ) in $V \subset E^N$ if it is both a subsolution and a supersolution in V. It is not difficult to see that a viscosity solution u satisfies (HJ) at each point $x \in V$ where the derivative $d_x u$ exists. We will prove the following.

PROPOSITION 14.

(1) *Any $u \in \mathcal{H}$ is differentiable almost everywhere, and is a viscosity subsolution of Hamilton–Jacobi in E^N.*

(2) *If $u \in \mathcal{H}$ is a fixed point of the Lax–Oleinik semigroup, then u is a viscosity solution of Hamilton–Jacobi.*

Proof. The fact that dominated functions are differentiable almost everywhere follows from Proposition 7, the fact that collisions are contained in a finite number of affine subspaces and Rademacher's theorem. In order to prove that they are viscosity subsolutions, take $u \in \mathcal{H}$ and $\psi : E^N \to \mathbb{R}$ of class C^1 such that $u - \psi$ admits a maximum at some $x_0 \in E^N$. Let $v \in E^N$. For all $t > 0$, we have

$$\psi(x_0) - \psi(x_0 - tv) \leq u(x_0) - u(x_0 - tv) \leq \frac{1}{2}\int_{-t}^{0} |v|^2\, ds + \int_{-t}^{0} U_\kappa(x_0 + sv)\, ds.$$

Dividing by t and taking the limit as $t \to 0$, we obtain

$$d_{x_0}\psi(v) \leq \tfrac{1}{2}|v|^2 + U_\kappa(x_0).$$

If we define $p_1, \ldots, p_N \in E$ by the condition $d_{x_0}\psi(v) = \sum_{i=1}^{N}\langle p_i, v_i\rangle_E$ for all $v = (v_1, \ldots, v_N) \in E^N$, then we can write

$$|d_{x_0}\psi|^2 = \sum_{i=1}^{N} m_i^{-1}\|p_i\|_E^2 = d_{x_0}\psi(w),$$

where $w = (w_1, \ldots, w_N)$ and $m_i w_i = p_i$ for all $i = 1, \ldots, N$. Therefore, using the last inequality with $v = w$, we obtain $|d_{x_0}\psi|^2 \le 2U_\kappa(x_0)$.

It remains to prove (2). Suppose that $u \in \mathcal{H}$ is such that $T_t^- u = u$ for all $t > 0$. Let $\psi : E^N \to \mathbb{R}$ be a C^1 function such that $u - \psi$ has a minimum at some $x_0 \in \Omega$.

With the same arguments as in the proof of Proposition 12, we deduce that there is a constant $k > 0$ such that $T_1^- u(x_0) = \inf\{u(y) + \phi(x_0, y, 1) \mid \|y - x_0\| \le k\}$. Therefore, using Theorem 1 and the lower semicontinuity of the Lagrangian action, we can choose $y_0 \in E^N$ such that $\|y_0 - x_0\| \le k$, and a curve $\gamma \in \mathcal{C}(x_0, y_0, 1)$ such that

$$u(x_0) = T_1^- u(x_0) = u(y_0) + \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2\, dt + \int_0^1 U_\kappa(\gamma(t))\, dt.$$

In particular, since u is a dominated function, we must have

$$u(x_0) - u(\gamma(t)) = \frac{1}{2}\int_0^t |\dot\gamma(s)|^2\, ds + \int_0^t U_\kappa(\gamma(s))\, ds$$

for all $t \in [0, 1]$, which means that γ is a calibrated curve for u. Using the fact that x_0 is a minimum for $u - \psi$, we conclude that

$$\psi(x_0) - \psi(\gamma(t)) \ge u(x_0) - u(\gamma(t)) = \frac{1}{2}\int_0^t |\dot\gamma(s)|^2\, ds + \int_0^t U_\kappa(\gamma(s))\, ds.$$

At this point we can prove that x_0 is a configuration without collisions. In other words, u is a viscosity supersolution at each collision configuration x_0 because there are no C^1 test functions ψ such that $u - \psi$ has a minimum in x_0. To see this, we proceed as follows.

Let $K > 0$ be a Lipschitz constant for the restriction of ψ to some neighborhood of x_0. Using the fact that $U_\kappa > 0$ in the last inequality, we deduce that

$$K|\gamma(t) - \gamma(0)| \ge \frac{1}{2}\int_0^t |\dot\gamma(s)|^2\, ds$$

for any sufficiently small $t > 0$. Applying the Cauchy–Schwarz inequality, we also have

$$t\int_0^t |\dot\gamma(s)|^2\, ds \ge \left(\int_0^t |\dot\gamma(s)|\, ds\right)^2 \ge |\gamma(t) - \gamma(0)|^2,$$

hence,

$$|\gamma(t) - \gamma(0)| \le 2Kt.$$

Again using the Lipschitz constant for ψ and the previous inequality, but this time neglecting the kinetic term, we obtain

$$2tK^2 \ge \int_0^t U_\kappa(\gamma(s))\, ds.$$

Dividing by t and taking the limit as $t \to 0$, we deduce that $U_\kappa(x_0) \le 2K^2$, meaning that $x_0 \in \Omega$. Since x_0 is a configuration without collisions, $\delta > 0$ exists such that $\gamma([0, \delta]) \subset \Omega$. Since γ is minimizing, hence a solution of the Euler–Lagrange flow in $[0, \delta]$, we know that γ is differentiable at $t = 0$. Dividing by t and taking the limit as $t \to 0$ in the inequality

$$\psi(x_0) - \psi(\gamma(t)) \ge \frac{1}{2}\int_0^t |\dot\gamma(s)|^2\, ds + \int_0^t U_\kappa(\gamma(s))\, ds,$$

we obtain

$$d_{x_0}\psi(v) \geq \tfrac{1}{2}|v|^2 + U_\kappa(x_0),$$

where $v = -\dot\gamma(0)$. On the other hand, we always have $2p(v) \leq |p|^2 + |v|^2$ for $p \in (E^*)^N$ and $v \in E^N$. Thus, we conclude that $|d_{x_0}\psi|^2 \geq 2U_\kappa(x_0)$. We have proved that u is a viscosity supersolution at x_0. □

3.4. *Lax–Oleinik and weak KAM solutions.* Following the analogy with the weak KAM theory for Tonelli Lagrangians on compact manifolds, we show that the fixed points of the Lax–Oleinik semigroup are the weak KAM solutions defined by Fathi in [**10**]. More precisely, we show that the fixed points of the Lax–Oleinik semigroup are characterized by the following property: given any configuration $x \in E^N$, we always have a calibrated curve $\gamma_x : (-\infty, 0] \to E^N$ such that $\gamma_x(0) = x$.

Recall that if $u : E^N \to \mathbb{R}$ is a dominated function, then a curve $\gamma : I \to E^N$ is said to be calibrated when it satisfies

$$u(\gamma(b)) - u(\gamma(a)) = A(\gamma|_{[a,b]})$$

for all compact intervals $[a, b] \subset I$. In particular, the calibrated curves of a dominated function are free time minimizers, meaning that

$$A(\gamma|_{[a,b]}) = \phi(\gamma(a), \gamma(b))$$

for all compact intervals $[a, b] \subset I$.

Therefore, the fixed points of the Lax–Oleinik semigroup can be characterized in terms of calibrated curves as follows (recall that our Lagrangian is symmetric).

PROPOSITION 15. *Let $u \in \mathcal{H}$ be a dominated function. Then, $u = T_t^- u$ for all $t > 0$ if and only if for each $x \in E^N$ there is a curve $\gamma_x : [0, +\infty) \to E^N$ with $\gamma_x(0) = x$ and such that $u(x) = u(\gamma_x(t)) + A(\gamma|_{[0,t]})$ for all $t > 0$.*

Proof. Suppose first that the condition is satisfied. Take $x \in E^N$ and the corresponding calibrated curve $\gamma_x : [0, +\infty) \to E^N$ with $\gamma_x(0) = x$. Since $u \in \mathcal{H}$, we already know that $u \leq T_t^- u$ for all $t > 0$. On the other hand, if we fix $t > 0$, we have $T_t^- u(x) \leq u(\gamma_x(t)) + A(\gamma|_{[0,t]}) = u(x)$. Therefore, u is a fixed point.

Suppose now that $u \in \mathcal{H}$ is a fixed point of (T_t^-). Given a configuration $x \in E^N$ and $t > 0$, we have

$$u(x) = T_t^- u(x) = \inf\{u(y) + \phi(x, y, t) \mid y \in E^N\}.$$

Using Lemma 13 and Theorem 2 (as in the proof of Proposition 12), we deduce that there is a constant $k > 0$ (depending on x and t) such that

$$u(x) = T_t^- u(x) = \inf\{u(y) + \phi(x, y, t) \mid y \in E^N \text{ and } \|y - x\| \leq k\}.$$

Therefore, using Theorem 1 and the lower semicontinuity of the Lagrangian action, we can choose $y(x, t) \in E^N$ such that $\|y(x, t) - x\| \leq k$, and a curve $\gamma_{x,t} \in \mathcal{C}(x, y(x, t), t)$ such that

$$u(x) = T_t^- u(x) = u(y(x, t)) + A(\gamma_{x,t}).$$

For each positive integer $n > 0$, we define the curve $\gamma_n : [0, n] \to E^N$ as the curve $\gamma_{x,n}$. Observe that if $m > n$, then $\gamma_m|_{[0,n]}$ minimizes the action in $\mathcal{C}(x, \gamma_m(n), n)$. Now we apply Theorem 1 and, once again, Lemma 13, and we deduce that for a fixed positive integer $n > 0$, the sequence $(A(\gamma_m|_{[0,n]}))_{m>n}$ is bounded. It is not difficult to see (using the Cauchy–Schwartz inequality) that an absolutely continuous curve $\gamma : I \to E^N$ with finite Lagrangian action must satisfy $|\gamma(t) - \gamma(s)| \leq 2A(\gamma)|t - s|^{1/2}$ for all $t, s \in I$. We can then apply Ascoli's theorem and deduce the existence of a convergent subsequence of $(\gamma_m|_{[0,n]})_{m>n}$. By a diagonal process, we can extract an increasing sequence of indexes $m_k \in \mathbb{N}$ such that, for each positive integer $n > 0$, the sequence $(\gamma_{m_k}|_{[0,n]})_{m_k>n}$ converges uniformly when $k \to \infty$. Observe now that, by construction, each curve $\gamma_{m_k}|_{[0,n]})_{m_k>n}$ calibrates the function u. The semicontinuity of the action therefore implies that the curve $\gamma_x : [0, +\infty) \to E^N$ defined by $\gamma_x(t) = \lim_{k\to\infty} \gamma_{m_k}(t)$ is also calibrated. □

We remark that for the Newtonian potential ($\kappa = 1/2$), Marchal's theorem implies—except of course in the collinear case ($\dim E = 1$)—that the calibrated curves of weak KAM solutions are true motions for $t > 0$ since they must be contained in Ω, the set of configurations without collisions. The dynamics of the free time minimizers of the Newtonian N-body problem is described in [**7**].

4. *The Kepler problem*

Unfortunately, the proof of the weak KAM theorem does not give any explicit solution. Nevertheless, we can give explicit solutions for the Hamilton–Jacobi equation of the Kepler problem.

We will first find isometry-invariant solutions when we have two bodies of unit mass in a line ($N = 2$, $m_1 = m_2 = 1$, $k = 1$) and a Newtonian potential ($\kappa = 1/2$). An invariant solution $u : \mathbb{R}^2 \to \mathbb{R}$ must satisfy $u(x + z, y + z) = u(x, y)$ and $u(x, y) = u(-x, -y)$ for all $x, y, z \in \mathbb{R}$. Therefore, the solution must be of the form $u(x, y) = f(|x - y|)$, and the Hamilton–Jacobi equation reads

$$u_x^2 + u_y^2 = 2|x - y|^{-1}.$$

Replacing $u(x, y)$ by $f(|x - y|)$, and solving the differential equation in f, we conclude that the unique invariant global solutions (up to an additive constant) are the functions

$$u_\pm(x, y) = \pm 2|x - y|^{1/2}.$$

In fact, the positive solution is the unique invariant fixed point of the *forward* Lax–Oleinik semigroup

$$T_t^+ u(x) = \inf\{u(y) - \phi(x, y, t) \mid y \in E^N\}.$$

The negative solution is therefore the unique invariant fixed point of the backward semigroup T_t^- (see Figure 3). Of course, since the Lagrangian is symmetric in speed, we have that $u \in \mathcal{H}$ is a backward solution if and only if $-u$ is a forward solution.

It is not difficult to see that we also have solutions invariant under translations, in particular the function b_+ given by $b_+(x, y) = u_-(x, y)$ for $x \geq y$ and $b_+(x, y) = u_+(x, y)$ for $x \leq y$ are also weak KAM solutions.

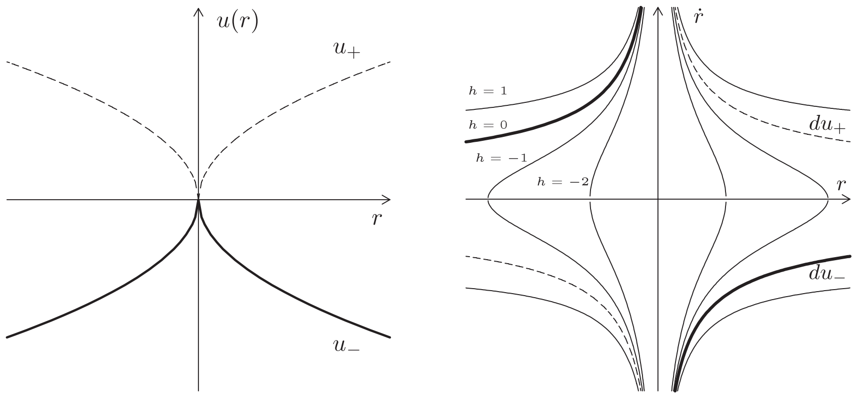

FIGURE 3. The two solutions and their derivatives for the one-dimensional Kepler problem.

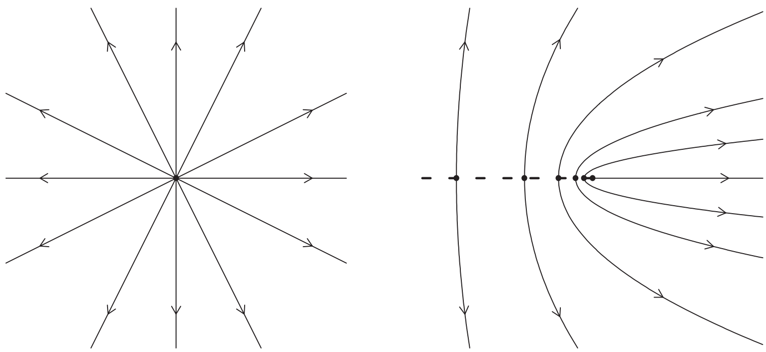

FIGURE 4. Calibrated curves of solutions for the planar Kepler problem.

For the planar Kepler problem, it is convenient first to reduce the problem by fixing the center of mass at the origin, or, equivalently, to look for translation-invariant solutions. Since the configuration is then determined by the position of the first body $x \in \mathbb{R}^2$, the problem reduces as usual to the fixed-center problem. If we denote by $x = (x_1, x_2)$ the position of the body, then the Hamilton–Jacobi equation reads

$$u_{x_1}^2(x) + u_{x_2}^2(x) = 2\|x\|^{-1}.$$

Undoubtedly, the simplest solution that we can give is the rotation-invariant solution

$$u(x_1, x_2) = -(x_1^2 + x_2^2)^{1/4}.$$

Its calibrated curves are all the parabolic homothetic motions, represented in the left-hand side of Figure 4. The half-parabolas on the right-hand side are the calibrated curves of a Buseman-type solution, which is constant and not differentiable over the dashed line. A computation made by Venturelli shows that this last solution can be explicitly defined by the formula

$$u(x_1, x_2) = -((x_1^2 + x_2^2)^{1/2} + x_1)^{1/2}.$$

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